UDC 62—50

ON THE EXISTENCE OF A SADDLE POINT IN A DIFFERENCE-DIFFERENTIAL ENCOUNTER-EVASION GAME

PMM Vol. 42, № 1, 1978, pp. 15-22

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(Received may 10, 1977)

A nonlinear difference-differential encounter-evasion game with a functional target is analyzed under integral constraints on the players' controls and functional constraints on segments of the controlled trajectories. Similarly to [1-3] a position procedure of control with a guide is constructed, solving the encounter and evasion problems. The existence of a saddle point in the game being analyzed is studied. The paper is closely related to the research in [1-9].

1. The following controlled system is specified:

$$x^{\cdot}(t) = f(t, x_t(s)) + F_1(t, x_t(s))\, u + F_2(t, x_t(s))\, v, \quad t_0 \leqslant t \leqslant \vartheta \tag{1.1}$$

Here x is the n-dimensional phase vector; u and v are the controls of the first and second players; the vector functional $f(t, x(s))$ and the matrix functionals $F_i(t, x(s))$, $i = 1, 2$, are determined on the set $[t_0, \vartheta] \times H_\omega$, where H_ω is the Hilbert space of n-dimensional functions $x(s)$ with the norm

$$\| x(s) \|_\omega = (\| x(0) \|^2 + \int_{-\omega}^{0} \| x(s) \|^2 ds)^{1/2}$$

$$\| z \| = (z_1^2 + \ldots + z_n^2)^{1/2}, \quad z \in E_n$$

and

$$f(t, x(s)) = f(t, x(-\tau_1), \ldots, x(-\tau_m), \varphi((t, x(s)))$$

where $\varphi(t, x(s))$ is a functional continuous on $[t_0, \vartheta]$, with values in E_r, satisfying (uniformly with respect to $t \in [t_0, \vartheta]$) a Lipschitz condition in $x(s)$ on each bounded set $D \subset H_\omega$, i.e.,

$$\| \varphi(t, x_1(s)) - \varphi(t, x_2(s)) \| \leqslant L \| x_1(s) - x_2(s) \|_\omega$$
$$L = L(D), \quad x_j(s) \in D, \quad j = 1, 2$$

The functions $f(t, z_1, \ldots, z_m, z)$ and $F_i(t, z), i = 1, 2$, are continuous in all the arguments and satisfy a Lipschitz condition in $(z_1, \ldots, z_m, z)$ and z, respectively. The growth conditions

$$\| f(t, x(s)) \| \leqslant \zeta_1(t) + \zeta_2(t) \| x(s) \|_\omega + \sum_{j=1}^{m} \eta_j(t) \| x(-\tau_j) \|$$

$$\| F_i(t, x(s)) \| \leqslant \zeta_{i+2}(t) + \varkappa_i \| x(s) \|_\omega$$

where $\zeta_i(t)$ and $\eta_j(t)$ are nonnegative square-summable functions and $\varkappa_i = \text{const} \geqslant 0$ are satisfied for any $x(s) \in H_\omega$. The control realizations $u[t]$ and $v[t]$ are subject to the constraints

$$\left(\int_{t_0}^{\infty} \| u[t] \|^2 dt\right)^{1/2} \leqslant \lambda[t_0], \quad \left(\int_{t_0}^{\infty} \| v[t] \|^2 dt\right)^{1/2} \leqslant \nu[t_0] \tag{1.2}$$

The changes in constraints $\lambda[t]$ and $\nu[t]$ are determined by the equalities

$$\lambda[t_2] = \lambda[t_1] - \left(\int_{t_1}^{t_2} \| u[t] \|^2 dt\right)^{1/2}$$

$$\nu[t_2] = \nu[t_1] - \left(\int_{t_1}^{t_2} \| v[t] \|^2 dt\right)^{1/2}$$

Let $\{u(\cdot);\ t_0,\ \vartheta;\ \lambda[t_0]\}$ and $\{v(\cdot);\ t_0,\ \vartheta;\ \nu[t_0]\}$ be summable functions on $[t_0, \vartheta]$, satisfying (1.2). The constraints on the right-hand side of system (1.1) guarantee the existence and continuability on $[t_0, \vartheta]$ of the solution of the Cauchy problem in the sense of Carathéodory for any initial $t_* \in [t_0, \vartheta]$ and $x(s) \in H_\omega$ and for any functions $u(t) \in \{u(\cdot); t_0, \vartheta;\ \lambda[t_0]\}$ and $v(t) \in \{v(\cdot);\ t_0, \vartheta; \nu[t_0]\}$. The unexplained concepts and notation below are contained in [9].

An element $x_0(s) \in H_\omega$ and the nonempty closed sets $N \subset [t_0, \vartheta] \times H_\omega$ and $M \subset [t_0 - \omega + \tau,\ \vartheta] \times H_\mu$ ($\mu = \text{const} \geqslant 0$, $\tau = \max \times [\omega, \mu]$) are specified. The encounter problem is to choose a feedback control u ensuring that the phase trajectory's segment $x[t+s;\ \mu]$ falls into $M(t)$ during the interval $[t_0 - \omega + \tau,\ \vartheta]$, leaving the segment $x[t+s;\ \omega]$ inside $N(t)$ for all $t \in [t_0, \vartheta]$. It is assumed that the first player can meet with any method of forming the control v developing measurable realizations $v[t]$ satisfying (1.2). the evasion problem is to choose a feedback control v ensuring that the segment $x[t+s;\ \mu]$ of phase trajectory $x[t]$ evades $M(t)$, leaving $x[t+s;\ \omega]$ inside $N(t)$ for all $t \in [t_0, \vartheta]$, or leading $x[t+s;\ \omega]$ out of $N(t)$ $(t_0 \leqslant t \leqslant \vartheta)$ before $x[t+s;\ \mu]$ falls into $M(t)$ $(t_0 - \omega + \tau \leqslant t \leqslant \vartheta)$. It is assumed as well that the second player, in his own turn, can meet with any method of forming the control u developing measurable on $[t_0, \vartheta]$ realizations $u[t]$ satisfying (1.2). Encounter and evasion games for conflict-controlled systems described by functional-differential equations under instantaneous constraints on the controls were analyzed in [3-5, 9]. The main difference between the present paper and those investigations is that here we study the case of integral constraints on the controls (see [2, 6-8].

2. We describe a procedure solving the encounter and evasion problems. The quadruple $p_{t_*} = \{t_*,\ \lambda_*,\ \nu_*,\ x_*(s;\ \tau)\}$ is called the game's position, R is the space of positions, $R^{(1)} = E_1 \times E_1 \times H_\tau$ and $p(t_*) = \{\lambda_*,\ \nu_*,\ x_*(s;\ \tau)\}$. The symbol $\sigma_\tau(p_{t_*},\ v(\cdot))$, $v(t) \in \{v(\cdot);\ t_*,\ \infty;\ \nu_*\}$, denotes the set of elements $p_t = \{t,\ \lambda(t),\ \nu(t),\ x(t+s;\ \tau)\}$ of the form

$$\vartheta \geqslant t \geqslant t_*, \quad \lambda^2(t) = \lambda_*^2 - J_u^2(t_*, t), \quad \nu^2(t) = \nu_*^2 - J_v^2(t_*, t)$$

$$x(t) = x_*(0;\ \tau) + \int_{t_*}^{t} [f(\xi, x_\xi(s)) + F_1(\xi, x_\xi(s))\, u(\xi) +$$
$$F_2(\xi, x_\xi(s))\, v(\xi)]\, d\xi$$

$$\left(J_u(t_*, t) = \left(\int_{t_*}^{t} \|u(\xi)\|^2 d\xi\right)^{1/2}, \quad J_v(t_*, t) = \left(\int_{t_*}^{t} \|v(\xi)\|^2 d\xi\right)^{1/2}\right)$$

$u(t)$ are all possible summable functions satisfying the inequality $J_u(t_*, \infty) \leqslant \lambda_*$. Let D be some set from R. We denote

$$D(t_*, t^*) = \{p_t = \{t, \lambda, \nu, x(s; \tau)\} \in D \mid t_* \leqslant t \leqslant t^*\}$$
$$D(t_*) = \{\{\lambda, \nu, x(s; \tau)\} \mid \{t_*, \lambda, \nu, x(s; \tau)\} \in D\}$$
$$D_\delta = \{\{t, \lambda, \nu, x(s; \delta)\} \mid \{t, \lambda, \nu, x(s; \tau)\} \in D, x(0; \delta) = x(0; \tau)$$
$$x(s; \delta) = x(s; \tau) \text{ for almost all } s \in [-\delta, 0]\} \quad (\delta \in [0, \tau])]$$
$$M^* = \{\{t, \lambda, \nu, x(s; \mu)\} \mid \{t, x(s; \mu)\} \in M, \lambda \geqslant 0, \nu \geqslant 0\}]$$
$$N^* = \{\{t, \lambda, \nu, x(s; \omega)\} \mid \{t, x(s; \omega)\} \in N, \lambda \geqslant 0, \nu \geqslant 0\}$$

The sets $W^{(u)}(t) \subset R^{(1)}$, $t_0 \leqslant t \leqslant \vartheta$, and $W_\omega^{(u)}(t) \subset N^*(t)$ are said to be u-stable if $W_\mu^{(u)}(\vartheta) \subset M^*(\vartheta)$ or $W^{(u)}(\vartheta) = \varnothing$ and for any $t_* \in [t_0, \vartheta)$, $t^* \in (t_*, \vartheta]$, $p(t_*) = \{\lambda_*, \nu_*, x_*(s; \tau)\} \in W^{(u)}(t_*)$ and $v(t) \in \{v(\cdot); t_*, \infty; \nu_*\}$ either $\sigma_\tau(t^*; p_{t_*}, v(\cdot)) \cap W^{(u)}(t^*) \neq \varnothing$ or $\sigma_\mu(p_{t_*}, v(\cdot)) \cap M^*(t_*, t^*) \neq \varnothing$. Here $\sigma_\tau(t^*; p_{t_*}, v(\cdot))$ is the section of set $\sigma_\tau(p_{t_*}, v(\cdot))$ by the hyperplane $t = t^*$

We introduce $u_*(p_{t_*}, p^*_{t_*}, \delta)$ and $v^*(p_{t_*}, p^*_{t_*}, \delta)$ $(\delta > 0)$ as the functions on which, respectively,

$$\min_{u(\cdot)} \left\{ \int_{t_*}^{t_*+\delta} b' u(t)\, dt \,\Big|\, \int_{t_*}^{t_*+\delta} \|u(t)\|^2 dt \leqslant \lambda^2 - \lambda^{*2} \right\} \quad \text{for} \quad \lambda > \lambda^*,\ b \neq 0$$

$$\max_{v(\cdot)} \left\{ \int_{t_*}^{t_*+\delta} c' v(t)\, dt \,\Big|\, \int_{t_*}^{t_*+\delta} \|v(t)\|^2 dt \leqslant \nu^{*2} - \nu^2 \right\} \quad \text{for} \quad \nu^* > \nu,\ c \neq 0$$

are achieved. Here

$$p_{t_*} = \{t_*, \lambda, \nu, x(s; \tau)\}, \quad p^*_{t_*} = \{t_*, \lambda^*, \nu^*, x^*(s; \tau)\}$$
$$b = (x(0; \tau) - x^*(0; \tau))' F_1(t_*, x(s; \tau))$$
$$c = (x(0; \tau) - x^*(0; \tau))' F_2(t_*, x(s; \tau))$$

(the prime denotes transposition). If $\lambda \leqslant \lambda^*$ or $b = 0$ $(\nu^* \leqslant \nu$ or $c = 0)$, we assume

$$u_*(p_{t_*}, p_{t_*}{}^*, \delta) = 0 \quad (v^*(p_{t_*}, p_{t_*}{}^*, \delta) = 0)$$

Let us define a procedure for the first player's control with the guide for specified initial position $p_{t_*} = \{t_0, \lambda[t_0], \nu[t_0], x_0(s; \tau)\}$ and u-stable sets $W^{(u)}(t)$, $t_0 \leqslant t \leqslant \vartheta$, $W^{(u)}(t_0) \neq \varnothing$. We take the element $p^*[t_0] = \{\lambda^*, \nu^*, x^*(s; \tau)\} \in W^{(u)}(t_0)$ closest to $p[t_0]$ (for simplicity we assume that such an element exists; the general case is investigated by passing to a minimizing sequence as was done in [4, 5]). Let Δ be a covering of interval $[t_0, \vartheta]$ by a system of half-open intervals

$$[\tau_i, \tau_{i+1}) \quad (i = 0, 1, \ldots, l(\Delta))$$
$$\tau_0 = t_0, \ \tau_l = \vartheta, \ \tau_{i+1} - \tau_i = \delta = \text{const}$$

We assume that in $[\tau_0, \tau_1)$ the motion of system (1.1) is generated by the constant control $u^{(0)}[t] = u_*(p_{t_0}, p_{t_0}^*, \delta)$ $(\tau_0 \leqslant t < \tau_1)$ in pair with some realization $v[t] \in \{v(\cdot); t_0, \infty; \nu[t_0]\}$. We then determine the position $p_{\tau_1} = \{\tau_1, \lambda[\tau_1], \nu[\tau_1], x[\tau_1 + s; \tau]\}$ at instant τ_1. We select the guide's position $p_{\tau_1}^*$ from the condition

$$p^*[\tau_1] \in W^{(u)}(\tau_1) \cap \sigma_\tau(\tau_1; p_{\tau_0}^*, v^{(0)}[\cdot])$$

$$(v^{(0)}[t] = v^*(p_{t_0}, p_{t_0}^*, \delta), \tau_0 \leqslant t < \tau_1)$$

assuming that this intersection is not empty. We define the first player's control on $[\tau_1, \tau_2)$ by the relation $u^{(1)}[t] = u_*(p_{\tau_1}, p_{\tau_1}^*, \delta)$ $(\tau_1 \leqslant t < \tau_2)$. The position p_{τ_2} is realized as a result of the choice of control $u^{(1)}[t]$ and of some control $v[t]$. We define the guide's position at instant $t = \tau_2$ from the condition

$$p^*[\tau_2] \in W^{(u)}(\tau_2) \cap \sigma_\tau(\tau_2; p_{\tau_1}^*, v^{(1)}[\cdot])$$

$$(v^{(1)}[t] = v^*(p_{\tau_1}, p_{\tau_1}^*, \delta), \tau_1 \leqslant t < \tau_2)$$

assuming once again that this intersection is not empty. If

$$W^{(u)}(\tau_{i+1}) \cap \sigma_\tau(\tau_{i+1}; p_{\tau_i}^*, v^{(i)}[\cdot]) \neq \varnothing \quad \text{for all} \quad i = 0, \ldots, l-1$$

we effect this procedure up to the instant $t = \vartheta$.

Let τ_j be the instant when first

$$W^{(u)}(\tau_j) \cap \sigma_\tau(\tau_j; p^*_{\tau_{j-1}}, v^{(j-1)}[\cdot]) = \varnothing$$

Then $M^*(\tau_{j-1}, \tau_j) \cap \sigma_\mu(p^*_{\tau_{j-1}}, v^{(j-1)}[\cdot]) \neq \varnothing$. Hence an instant $\tau_* \in [\tau_{j-1}, \tau_j]$ exists when the guide's position $p_{\tau_*}^* = \{\tau_*, \lambda_*, \nu_*, x[\tau_* + s; \tau]\}$ can be determined from the conditions

$$p_{\tau_*}^* \in \sigma_\tau(p^*_{\tau_{j-1}}, v^{(j-1)}[\cdot])$$

$$\{\lambda_*, \nu_*, x[t_* + s; \mu]\} \in M(\tau_*)$$

At instant $t = \tau_j$ we take an arbitrary element from $\tau_j \times \sigma_\tau(\tau_j; p_{\tau_*}^*, v^{j-1}[\cdot])$ as the guide's position $p_{\tau_j}^*$. Further, we define the controls $u^{(i)}[t]$ and $v^{(i)}[t]$ $(\tau_i \leqslant t < \tau_{i+1}, j \leqslant i \leqslant l-1)$ by the relations

$$u^{(i)}[t] = u_*(p_{\tau_i}, p_{\tau_i}^*, \delta), \quad v^{(i)}[t] = v^*(p_{\tau_i}, p_{\tau_i}^*, \delta)$$

and we choose the guide's position arbitrarily from the sets $\tau_{i+1} \times \sigma_\tau(\tau_{i+1}; p_{\tau_i}^*, v^{(i)}[\cdot])$. The motion constructed of system (1.1) is denoted

$$x_\Delta[t] = x[t; p_{t_0}, u_\delta, v]$$

$$u_\delta[t] = u^{(i)}[t], \quad \tau_i \leqslant t < \tau_{i+1}, \qquad i = 0 \ldots, l-1$$

Function $x[t]$, $t_0 \leqslant t \leqslant \vartheta$, is called a motion of system (1.1) if there exists a sequence of functions $x_{\Delta_k}[t] = x[t; p_{t_0}^{(k)}, u_{\delta_k}, v_k]$ satisfying the conditions

$$x_{\Delta_k}[t] \to x[t] \text{ in } C([t_0, \vartheta]) \tag{2.1}$$

$$(\tau_{i+1}(k) - \tau_i(k)) \to 0$$

$$p_{t_0}^{(k)} \to p_{t_0} \quad \text{as} \quad k \to \infty$$
$$p_{t_0}^{(k)} = \{t_0, \lambda_0^{(k)}, \nu_0^{(k)}, x_0^{(k)}(s; \tau)\}$$

It can be shown that this motion (we denote it $x[t;\ p_{t_0},\ W^{(u)}]$) exists. Without loss of generality we assume $M(\vartheta) \neq \varnothing$.

Lemma 2.1. If u-stable sets $W^{(u)}(t)$, $t_0 \leqslant t \leqslant \vartheta$, exist such that $p[t_0] \in W^{(u)}(t_0)$ and $W_\mu^{(u)}(\vartheta) \subset M^*(\vartheta)$, then for any motion $x[t] = x[t; p_{t_0}, W^{(u)}]$ we can find an instant $t_* \in [t_0 - \omega + \tau, \vartheta]$ when first $\{t_*, x[t_* + s; \mu]\} \in M$, and $\{t, x[t+s; \omega]\} \in N$ for $t \in [t_0, t_*]$.

We present the lemma's proof. Let Δ_k be a covering of interval $[t_0, \vartheta]$ by the intervals $\tau_i(k) \leqslant t < \tau_{i+1}(k)$, $i = 0, \ldots, l_k$, $\tau_0(k) = t_0$, $\tau_{l_k} = \vartheta$, $l_k = l(\Delta_k)$; let $x_{\Delta_k}[t] = x_{\Delta_k}[t;\ p_{t_0}^{(k)},\ u_{\delta_k},\ v_k]$ be the phase vector of system (1.1) realized at instant t; let $x_{\Delta_k}^*[t]$ be the phase vector of the guide, whose motion was formed jointly with motion $x_{\Delta_k}[t]$; let $u_{\delta_k}^*[t]$ be the first player's control whose action realizes motion $x_{\Delta_k}^*[t]$.

It can be verified that the equality

$$\lim_{m\to\infty} \sum_{i=0}^{m-1} \left(\int_{i\zeta}^{(i+1)\zeta} \|\varphi(t)\| dt \right)^2 = 0, \quad \zeta = \frac{\vartheta - t_0}{m} \tag{2.2}$$

is valid for any n-dimensional vector function $\varphi(t) \in L^2[t_0, \vartheta]$. Proceeding from the method of forming motions $x_{\Delta_k}[t]$ and $x_{\Delta_k}^*[t]$, using (2.2) we establish the relation

$$\lim_{k\to\infty} \max_i \{\|r_{k,i}\|,\ i = 0, \ldots, l_k\} = 0 \tag{2.3}$$

Here

$$r_{k,i} = \|x_{\Delta_k}[\tau_i(k) + s; \tau] - x_{\Delta_k}^*[\tau_i(k) + s; \tau]\|_\tau +$$
$$\sum_{j=1}^{m} \int_{-\tau_j}^{0} \|x_{\Delta_k}[\tau_i(k) + s; \tau] - x_{\Delta_k}^*[\tau_i(k) + s; \tau]\| ds +$$
$$2(x_{\Delta_k}[\tau_i(k)] - x_{\Delta_k}^*[\tau_i(k)])' \left(\int_{\tau_{i-1}(k)}^{\tau_i(k)} F_1(\xi, x_{\Delta_k}^*[\xi + s]\right) u_{\delta_k}^*[\xi] d\xi -$$
$$\int_{\tau_{i-1}(k)}^{\tau_i(k)} F_2(\xi, x_{\Delta_k}[\xi + s]) v_k[\xi] d\xi$$

The lemma's validity follows from (2.3).

Let
$$\Phi^* = \{p_t = \{t, \lambda, \nu, x(s; \tau)\} \mid \{t, x(s; \tau)\} \in \Phi,\ \lambda \geqslant 0,\ \nu \geqslant 0\}$$
$$\Psi^* = \{p_t = \{t, \lambda, \nu, x(s; \tau)\} \mid \{t, x(s; \tau)\} \in \Psi,\ \lambda \geqslant 0,\ \nu \geqslant 0\}$$

where Φ and Ψ are closed sets in $E_1 \times H_\tau$, satisfying the conditions $(\alpha > 0)$

$$\overline{\Phi}_\mu^\alpha \cap M = \varnothing, \quad \overline{\Psi}_\omega^\alpha \cap N = \varnothing$$

The sets $W^{(v)}(t) \subset R^{(1)}$, $t_0 \leqslant t \leqslant \vartheta$, $W^{(v)}(t) \subset \Phi^*(t)$, are said to be v-stable if for any $t_* \in [t_0, \vartheta)$, $t^* \in (t_*, \vartheta]$, $p(t_*) = \{\lambda_*, \nu_*, x_*(s; \tau)\} \in W^{(v)}(t_*)$

and $u(t) \in \{u(\cdot);\ t_*,\ \infty;\ \lambda_*\}$ either $\sigma_\tau(t^*;\ p_{t_*},\ u(\cdot)) \cap W^{(v)}(t^*) \neq \varnothing$ or $\sigma_\tau(p_{t_*}, u(\cdot)) \cap \Psi^*(t_*, t^*) \neq \varnothing$.

We introduce $u^*(p_{t_*}, p_{t_*}^*, \delta)$ and $v_*(p_{t_*}, p_{t_*}^*, \delta)$ as the functions on which, respectively,

$$\max_{u(\cdot)} \left\{ \int_{t_*}^{t_*+\delta} b'u(t)\,dt \mid \int_{t_*}^{t_*+\delta} \|u(t)\|^2\,dt \leqslant \lambda^{*2} - \lambda^2 \right\} \quad \text{for } \lambda^* > \lambda,\ b \neq 0$$

$$\min_{v(\cdot)} \left\{ \int_{t_*}^{t_*+\delta} c'v(t)\,dt \mid \int_{t_*}^{t_*+\delta} \|v(t)\|^2\,dt \leqslant \nu^2 - \nu^{*2} \right\} \quad \text{for } \nu > \nu^*,\ c \neq 0$$

are achieved. If $\lambda \geqslant \lambda^*$ or $b = 0$ $(\nu^* \geqslant \nu$ or $c = 0)$, then we assume $u^*(p_{t_*}, p_{t_*}^*, \delta) = 0$ $(v_*(p_{t_*}, p_{t_*}^*, \delta) = 0)$.

Let us define a procedure for the second player's control with the guide for v-stable sets $W^{(v)}(t)$, $t_0 \leqslant t \leqslant \vartheta$. We form the second player's control as follows:

$$v_\delta[t] = v_*(p_{\tau_i}, p_{\tau_i}^*, \delta) \quad (\tau_i \leqslant t < \tau_{i+1},\ i = 0, \ldots, l-1)$$

Here p_{τ_i} is the game's position and $p_{\tau_i}^*$ is the guide's position realized at instant $t = \tau_i$. The controls

$$u_\delta^*[t] = u^*(p_{\tau_i}, p_{\tau_i}^*, \delta) = u^{(i)}[t], \quad \tau_i \leqslant t < \tau_{i+1}$$

are used to determine the guide's positions. As the guide's initial position $p_{t_0}^*$ we select the element of set $t_0 \times W^{(v)}(t_0)$ closest to p_{t_0} (once again we assume the existence of such an element). Next, we determine the positions $p_{\tau_i}^*$ successively from the condition

$$p^*[\tau_i] \in \sigma_\tau(\tau_i;\ p^*_{\tau_{i-1}},\ u^{(i-1)}[\cdot]) \cap W^{(v)}(\tau_i)$$

either up to the instant $\tau_l = \vartheta$ if all these intersections are nonempty or up to the instant τ_j for which this intersection first is empty. The position $p_{\tau_j}^*$ at instant τ_j is determined from the condition

$$p^*[\tau_j] \in \sigma_\tau(\tau_j;\ p_{\tau_*}^*,\ u^{(j-1)}[\cdot])$$

where $p^*[\tau_*] \in \sigma_\tau(\tau_*;\ p^*_{\tau_{j-1}},\ u^{(j-1)}[\cdot]) \cap \Psi^*(\tau_*)$, $\tau_{j-1} \leqslant \tau_* \leqslant \tau_j$. The existence of such an element $p_{\tau_*}^*$ follows from the inclusion $p^*[\tau_{j-1}] \in W^{(v)}(\tau_{j-1})$ and from the definition of v-stability of sets $W^{(v)}(t)$. Next, as $p_{\tau_i}^*$ $(j < i \leqslant l)$ we choose arbitrary elements from the sets $\tau_i \times \sigma_\tau(\tau_i;\ p^*_{\tau_{i-1}},\ u^{(i-1)}[\cdot])$.

By $x_\Delta[t] = x[t;\ p_{t_0},\ u,\ v_\delta]$ we denote the motion of system (1.1), realized by the second-player's control $v_\delta[t]$, $t_0 \leqslant t \leqslant \vartheta$, in pair with some control $u[t] \in \{u(\cdot);\ t_0,\ \infty;\ \lambda[t_0]\}$. By $x[t;\ p_{t_0},\ W^{(v)}]$ we denote the function $x[t]$, $t_0 \leqslant t \leqslant \vartheta$ generated by the sequence of motions $x_{\Delta_k}[t] = x[t; p_{t_0}^{(k)},\ u_k,\ v_{\delta_k}]$ satisfying (2.1).

Analogously to Lemma 2.1 we can prove

Lemma 2.2. If v-stable sets $W^{(v)}(t)$, $t_0 \leqslant t \leqslant \vartheta$ exist such that $p[t_0] \in W^{(v)}(t_0)$, then for any motion $x[t] = x[t;\ p_{t_0},\ W^{(v)}]$ the element $\{t, x[t+s;\ \tau]\}$ remains in domain Φ up to instant ϑ or up to the instant τ_* when first $\{\tau_*,\ x[\tau_* + s;\ \tau]\} \in \Psi$.

Let $W_*^{(v)}(t)$, $t_0 \leqslant t \leqslant \vartheta$, be a maximal v-stable system of sets.

We denote $W_*^{(u)}(t) = R^{(1)} \setminus W_*^{(v)}(t)$.

Theorem 2.1. For any initial game position p_{t_0} :either $p[t_0] \in W_*^{(u)}(t_0)$, and then the encounter problem has a solution which is provided by a procedure of control with a guide defined for u-stable sets $W_*^{(u)}(t)$, $t_0 \leqslant t \leqslant \vartheta$, or $p[t_0] \notin W_*^{(u)}(t_0)$, and then the evasion problem has a solution, it being that $p[t_0] \in W^{(v)}(t_0)$, where $W^{(v)}(t)$, $t_0 \leqslant t \leqslant \vartheta$, are certain v-stable sets, and this solution is provided by a procedure of control with a guide, defined for the sets $W^{(v)}(t)$, $t_0 \leqslant t \leqslant \vartheta$.

The theorem can be proved along the plan of the proof of Theorem 3.3 in [2].

3. Let us assume that $N = [t_0, \vartheta] \times H_\omega$ and that the first player measures the phase states of system (1.1) inaccurately. That is, at the instant t he knows the quantity $w[t+s;\tau]$ connected with the realization $x[t+s;\tau]$ by the relation

$$\| w[t+s;\tau] - x[t+s;\tau] \|_\tau \leqslant \alpha, \quad t_0 \leqslant t \leqslant \vartheta, \ \alpha = \text{const} \geqslant 0$$

We define the motion $x_0[t; p_{t_0}, W^{(u)}]$ similarly to motion $x[t; p_{t_0}, W^{(u)}]$. The difference is that we equate the controls $u^{(i)}[t]$ and $v^{(i)}[t]$ when $t \in [\tau_i, \tau_{i+1})$

$$u^{(i)}[t] = u_*(p_{\tau_i}^{(0)}, p_{\tau_i}^*, \delta), \quad v^{(i)}[t] = v^*(p_{\tau_i}^{(0)}, p_{\tau_i}^*, \delta)$$

$$p_t^{(0)} = \{t, \lambda[t], \nu[t], w[t+s;\tau]\}$$

Lemma 3.1. Let closed u-stable sets $W^{(u)}(t) \subset R^{(1)}$, $t_0 \leqslant t \leqslant \vartheta$, exist such that $p[t_0] \in W^{(u)}(t_0)$ and $W_\mu^{(u)}(\vartheta) \subset M^*(\vartheta)$. Then, for any number $\varepsilon > 0$ there exists a number $\alpha > 0$ such that for any motion $x_0[t; p_{t_0}, W^{(v)}]$ we can find an instant $t_* \in [t_0 - \omega + \tau, \vartheta]$ when first $\{t_*, x[t_* + s; \mu]\} \in \bar{M}^\varepsilon$.

The lemma's proof is analogous to that of Lemma 2.1.

We note that when $N = [t_0, \vartheta] \times H_\omega$ the condition

$$\sigma_\tau(p_{t_*}, u(\cdot)) \cap \Psi^*(t_*, t^*) \neq \varnothing$$

should be dropped in the definition of the v-stability of sets $W^{(v)}(t)$, $t_0 \leqslant t \leqslant \vartheta$. If at instant t the second player also knows the quantity $w[t+s;\tau]$, then we can define motion $x_0[t; p_{t_0}, W^{(v)}]$ similarly to motion $x[t; p_{t_0}, W^{(v)}]$ by setting

$$u_\delta^*[t] = u^*(p_{\tau_i}^{(0)}, p_{\tau_i}^*, \delta), \quad v_\delta[t] = v_*(p_{\tau_i}^{(0)}, p_{\tau_i}^*, \delta)$$

for $t \in [\tau_i, \tau_{i+1})$

Then there holds

Lemma 3.2. Let v-stable sets $W^{(v)}(t)$, $t_0 \leqslant t \leqslant \vartheta$, and $p[t_0] \in W^{(v)}(t_0)$ be specified. Numbers $\varepsilon > 0$ and $\alpha_0 > 0$ exist such that the condition

$$x_0[t+s;\mu] \notin \bar{M}^\varepsilon(t), \quad t_0 - \omega + \tau \leqslant t \leqslant \vartheta$$

is specified for the motions $x_0[t; p_{t_0}, W^{(v)}]$ if $\alpha \leqslant \alpha_0$.

Suppose that by choosing a control $u[t]$ the first player strives to minimize the value of some continuous functional $\varphi(x(s;\mu))$ at the instant ϑ, while choosing a control $v[t]$ the second player strives to maximize at instant ϑ the value of $\varphi(x(s;\mu))$ on the trajectories of system (1.1). The functional $\varphi(x(s;\mu))$ is defined on space H_μ.

Relying on Theorem 2.1, just as in [1] (see Sect. 18,97) we can validate

Theorem 3.1. For any initial position p_{t_0} a number c_0, u-stable sets $W^{(u)}(t)$, $t_0 \leqslant t \leqslant \vartheta$, and v-stable sets $W^{(v)}(t)$, $t_0 \leqslant t \leqslant \vartheta$ exist such that the relation

$$\varphi(x[\vartheta + s;\ p_{t_0},\ W^{(u)}]) \leqslant c_0 \leqslant \varphi(x[\vartheta + s;\ p_{t_0},\ W^{(v)}])$$

holds.

The author thanks Iu. S. Osipov for posing the problem and for valuable advice.

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Translated by N. H. C.